Math 10B with Professor Stankova
Worksheet, Discussion #7; Thursday, 2/8/2018
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1 Induction

1.1 Examples

1. Prove that $5^{2n+1} + 2^{2n+1}$ is divisible by 7 for all $n \geq 0$.

Solution: First we show the base case. We have that $5^{2\cdot 0+1} + 2^{2\cdot 0+1} = 5 + 2 = 7$, which is divisible by 7. Now assume the inductive hypothesis that $5^{2n+1}+2^{2n+1}$ is true for some $n \geq 0$. Then $5^{2(n+1)+1}+2^{2(n+1)+1} = 5^{2n+1+2}+2^{2n+1+2} = 25\cdot 5^{2n+1}+4\cdot 2^{2n+1} = 21 \cdot 5^{2n+1} + 4(5^{2n+1} + 2^{2n+1})$. The former is divisible by 7 and so is the latter which means the sum is. Thus, by mathematical induction, the result holds for all $n \geq 0$.

2. Find a formula for

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \cdots + \frac{1}{(2n-1)(2n+1)}$$

for $n \geq 1$

Solution: We try some small values. We see the results are $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \cdots$. So we guess that the formula is $\frac{n}{2n+1}$. First we show the base case which is true since $\frac{1}{1\times 3} = \frac{1}{3}$. Now we assume the inductive hypothesis that the result is true for n. Now for $n+1$, we have that

$$\left(\frac{1}{1\times 3} + \frac{1}{3\times 5} + \cdots + \frac{1}{(2n-1)(2n+1)}\right) + \frac{1}{(2n+1)(2n+3)} \overset{IH}{=} \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)}$$

$$= \frac{2n^2+3n+1}{(2n+1)(2n+3)} = \frac{(2n+1)(n+1)}{(2n+1)(2n+3)} = \frac{n+1}{2n+3},$$

as required. Therefore, by mathematical induction, we have proven the result for all $n \geq 1$.

1.2 Problems

3. **TRUE** False If we want to prove S_n for all $n \geq 10$, then our base case would be $n = 10$.

4. True **FALSE** When using induction, if we can show that if S_{100} is true, then S_{101} is true, then S_n must be true for all n.

> **Solution:** When doing the inductive step, we must use a general n, not a specific case (although a specific case when help you find a pattern)

5. **TRUE** False Instead of assuming S_n is true and showing that S_{n+1} is true, we can instead assume that S_{n-1} is true and prove that S_n is true.

> **Solution:** This is effectively the same thing in showing that if one day it rains, then it rains the next.

6. Prove that $n^3 + 2n$ is divisible by 3 for all integers $n \geq 0$.

> **Solution:** We show that it is true for the base case $n = 0$. This is true because 0 is divisible by 3. Now we assume the inductive hypothesis that $n^3 + 2n$ is divisible by 3. Now, we have that
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> $$(n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 1 = (n^3 + 2n) + 3(n^2 + n + 1)$$
>
> which is divisible by 3 by the inductive hypothesis. Therefore, we have shown the result by mathematical induction.

7. Find a formula for

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)}.$$

> **Solution:** We write out the first few terms as $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$ and guess that the answer is $\frac{n}{n+1}$. This is true for the base case. Assuming the inductive hypothesis, we have that
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> $$\left(\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)}\right) + \frac{1}{(n+1)(n+2)} \overset{IH}{=} \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$
>
> $$= \frac{n^2 + 2n + 1}{(n+1)(n+2)} = \frac{n+1}{n+2}.$$

8. Prove that for $n \geq 1$

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.$$

Solution: This is true for the base case and assuming true for general n, we have that

$$(1 + 4 + \cdots + (3n - 2)) + (3(n + 1) - 2) \stackrel{IH}{=} \frac{n(3n-1)}{2} + (3n + 1) = \frac{(n+1)(3n+2)}{2}.$$

9. Prove that $6^n - 1$ is divisible by 5 for $n \geq 1$.

Solution: This is true for the base case $n = 1$. Assuming true for n, we have that $6^{n+1} - 1 = 6 \cdot 6^n - 1 = 5 \cdot 6^n + (6^n - 1)$. The former is divisible by 5 and the latter is by the inductive hypothesis, therefore the whole thing is divisible by 5. Thus, by mathematical induction, the result is shown.

10. Prove that $1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \cdots + n! \cdot n = (n + 1)! - 1.$ for all $n \geq 1$.

Solution: The result is true for $n = 1$ and assuming true for general n, we have that

$$1! \cdot 1 + \cdots + n! \cdot n + (n+1)! \cdot (n+1) = (n+1)! - 1 + (n+1)!(n+1) = (n+1)!(n+2) - 1 = (n+2)! - 1.$$

Thus by mathematical induction, the result is proven.

11. Prove that $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n + 1) = \frac{n(n+1)(n+2)}{3}$ for all $n \geq 1$.

Solution: The result is true for $n = 1$. Now assume true for some n. Then

$$1 \times 2 + \cdots + n \times (n + 1) + (n + 1)(n + 2) = \frac{n(n+1)(n+2)}{3} + (n + 1)(n + 2)$$

$$= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} = \frac{(n+3)(n+1)(n+2)}{3}.$$

Thus by mathematical induction, the result is true.

12. (Challenge) Let the Fibonacci numbers be defined as $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$. Then show that
$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Solution: Look online or come to office hours if you want help with the computation. The main thing is here our base case is both $n = 1$ and $n = 2$.

2 Probability

2.1 Examples

13. Assume that a telephone number is a 7 digit number that does not begin with 0 or 1. If I pick a random telephone number, what is the probability that it begins with a 9 or ends with a 0? What is the probability space?

Solution: The probability space is all telephone numbers and the event is the subset of telephone numbers that begin with 9 or end with 0. We solve this using PIE by saying that the probability is the probability that it begins with 9, plus the probability that it ends with 0, minus the probability that both are true. The probability that it begins with 9 is $\frac{10^6}{8\cdot 10^6} = \frac{1}{8}$. The probability that it ends with 0 is $\frac{8\cdot 10^5}{8\cdot 10^6} = \frac{1}{10}$. The probability of both being true is $\frac{10^5}{8\cdot 10^6} = \frac{1}{80}$. Thus, the probability is
$$\frac{1}{8} + \frac{1}{10} - \frac{1}{80} = \frac{17}{80}.$$

14. In the 3 classes I'm taking, each has 3 HW assignments that have to be done in order (for a total of 9 HW assignments). If I randomly pick an order to do these 9 HWs, what is the probability that I actually do each of the 3 HWs in order?

Solution: There are 9! ways to pick in what order I do the homework. Then, in order to find out the number of ways there are to do them in order, first choose the order in which to do the homework if I only care about which class's work I am doing. There are $\binom{9}{3}\cdot\binom{6}{3}\binom{3}{3}$ ways to do this and for each of these ways, there is only 1 way to order the homework for the class so I do them in order. Thus, there are $\binom{9}{3,3,3}$ ways to do these HWs in order. Thus, the probability that I do them in order is $\frac{\binom{9}{3}\binom{6}{3}\binom{3}{3}}{9!} = \frac{1}{3!3!3!} = \frac{1}{216}$.

2.2 Problems

15. True **FALSE** The probability function P takes outcomes and outputs a probability for that outcome.

> **Solution:** The probability function takes in events, which are subsets of the probability space and a set of outcomes, and outputs a probability for that event.

16. True **FALSE** When calculating the probability of an event $A \subset \Omega$, we can always take $P(A) = |A|/|\Omega|$.

> **Solution:** This is only true if all the outcomes are equally likely. We assume that for these problems but later on we'll see cases when that isn't necessarily true (for example, in a biased coin the probability of heads and tails are no longer 50%.)

17. I roll 4 6-sided die. What is the probability that the sum of the numbers rolled is 7?

> **Solution:** Let the die be distinct. Then there are 6^4 total die rolls. The number of ways to have the sum be 6 is if $x_1 + x_2 + x_3 + x_4 = 7$, but all numbers are greater than 1 so we can subtract 4 to get $x_1 + x_2 + x_3 + x_4 = 3$ and no restrictions. This is a balls and urns problem and this can be done in $\binom{3+4-1}{3} = \frac{6}{3}$ ways. Thus, the probability is $\frac{\binom{6}{3}}{6^4}$.

18. I am giving out grades to 60 students randomly (A, B, C, D, F). What is the probability that at least half the class got A's?

> **Solution:** Let x_A, x_B, x_C, x_D, x_F be the number of students that got each grade. Then $x_A + x_B + x_C + x_D + x_F = 60$ and the total number of ways to assign grades is $\binom{60+5-1}{60} = \binom{64}{60}$ ways. The number of ways that half the class got A's is $x_A \geq 30$ so we give out 30 A's and then do the problem again with 30 students to get $\binom{30+5-1}{30} = \binom{34}{30}$ ways. So the probability is
> $$\frac{\binom{34}{30}}{\binom{64}{60}}.$$

19. Out of the 14 pants that I own, there are 5 of them that are white. Every day for two weeks, I randomly put on a new pair of pants. What is the probability that I won't wear white pants two days in a row?

Solution: This is like the book marking problem. Since I only care about which pants are white, we can consider all the pants that are white the same and the ones not white all the same. Thus, there are a total of $\binom{14}{5}$ ways to choose an ordering of pants. Then to count the number of ways to do this so that I don't wear white pants two days in a row, I first put down the non-white pants and in the 10 spots next to these 9 pants, I choose 5 of them to wear white pants. Thus, there are $\binom{10}{5}$ ways to do this. This gives a probability of

$$\frac{\binom{10}{5}}{\binom{14}{5}}.$$

Note that if you didn't consider the pants distinguishable, you would get the same answer. This is because there are 14! ways to order which order I will wear the pants. Then once we choose an order of white and nonwhite pants, there are 5! ways to rearrange the white pants and 9! ways to rearrange the non-white pants. Thus, the probability is $\frac{5!9!\binom{10}{5}}{14!}$, which is the same.

20. I pick a random 5 digit number. What is the probability that its digits are in increasing (could be repeating) order?

Solution: There are a total of $9 \cdot 10^4$ total 5 digit numbers. If the digits are increasing, then no digit can be 0 and any collection of 5 numbers with repeats from 1 through 9 has one way to arrange them in an increasing order. Thus, there are a total of $\binom{5+9-1}{5} = \binom{13}{5}$ different 5 digit numbers whose digits are in increasing order. Thus, the probability of choosing one of these is $\frac{\binom{13}{5}}{9 \cdot 10^4}$.

21. I roll 5 die for Yahtzee. What is the probability that I get a 5 in a row?

Solution: There are going to be 6^5 different possibilities for die rolls. Then to get 5 in a row, I have to get $1-5$ or $2-6$ and for each of these options, there are 5! ways to have the 5 die get those numbers. Thus, the probability is $\frac{2 \cdot 5!}{6^5}$.

22. I pick 3 numbers a, b, c (not necessarily different) numbers from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. What is the probability that $ab + c$ is even?

Solution: The probability that ab is odd is if we choose odd numbers for both a and b so $\frac{5}{10} \cdot \frac{5}{10} = \frac{1}{4}$. The probability that it is even is 1 minus that or $\frac{3}{4}$. The probability that c is odd or even is both $\frac{1}{2}$. So the probability of $ab + c$ being even is either when ab, c are both odd, or if they are both even. The first occurs with probability $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$ and the latter with probability $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$ giving a total of $\frac{1}{8} + \frac{3}{8} = \frac{1}{2}$.